

The reverse order law of the (b, c) -inverse in rings

Yuanyuan Ke^{a*}, Dijana Mosić^{b†}, Jianlong Chen^{a‡}

^aDepartment of Mathematics, Southeast University, Nanjing 210096, China

^bFaculty of Sciences and Mathematics, University of Niš, Višegradska 33,
18000 Niš, Serbia

Abstract: We present equivalent conditions of reverse order law for the (b, c) -inverse $(aw)^{(b,c)} = w^{(b,s)}a^{(t,c)}$ to hold in a ring. Also, we study various mixed-type reverse order laws for the (b, c) -inverse. As a consequence, we get results related to the reverse order law for the inverse along an element. More general case of reverse order law $(a_1a_2)^{(b_3,c_3)} = a_2^{(b_2,c_2)}a_1^{(b_1,c_1)}$ is considered too.

Keywords: Generalized inverse; (b, c) -inverse; the inverse along an element; ring.

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1 Introduction

Throughout the paper, we assume that R is a ring with identity. We use R^\bullet to denote the set of all idempotents ($p^2 = p$) of R .

Let $b, c \in R$. The concept of the (b, c) -inverse as a generalization of the Moore-Penrose inverse, the Drazin inverse, the Chipman's weighted inverse and the Bott-Duffin inverse, was for the first time introduced by Drazin in 2012 [4] in the settings of rings. Recall that an element $a \in R$ is said to be (b, c) -invertible if there exists $y \in R$ such that

$$y \in (bRy) \cap (yRc), \quad yab = b, \quad cay = c.$$

If such y exists, it is unique and it is called the (b, c) -inverse of a , denoted by $a^{(b,c)}$. The set of all (b, c) -invertible elements of R will be denoted by $R^{(b,c)}$. For more results of the (b, c) -inverse, we refer the reader to see [5, 6, 16, 24].

Drazin [4] introduced an outer generalized inverse relative to a pair of idempotents $e, f \in R$ which intermediates between the Bott-Duffin inverse and the (b, c) -inverse. This generalized

*E-mail: keyy086@126.com

†Corresponding author. E-mail: dijana@pmf.ni.ac.rs

‡E-mail: jlchen@seu.edu.cn

inverse is called the Bott-Duffin (e, f) -inverse. Recall that the Bott-Duffin (e, f) -inverse of $a \in R$ is the element $y \in R$ which satisfies

$$y = ey = yf, \quad yae = e, \quad fay = f.$$

If the Bott-Duffin (e, f) -inverse of a exists, it is unique and denoted by $a_{e,f}^{BD}$. According to Kantún-Montiel [14, Proposition 3.4], an image-kernel (e, f) -inverse is the Bott-Duffin $(e, 1-f)$ -inverse. More results on the Bott-Duffin (e, f) -inverse and image-kernel (p, q) -inverse can be found in [15, 20, 22, 23].

Mary in [18] introduced a new generalized inverse, called the inverse along element. An element $a \in R$ is said to be invertible along $d \in R$ (or Mary invertible) if there exists $y \in R$ such that

$$yad = d = day, \quad yR \subseteq dR, \quad Ry \subseteq Rd.$$

If such $y \in R$ exists, it is unique and will be denoted by $a^{\parallel d}$. We use $R^{\parallel d}$ to denote the all Mary invertible elements in R . This inverse unify some well-known generalized inverses, such as the group inverse, Drazin inverse and Moore-Penrose inverse. Also, the inverse along element d is a special case of (b, c) -inverse, for $(b, c) = (d, d)$ [4, Proposition 6.1]. Several authors also have studied this new outer inverse (see [1, 2, 19, 25, 27]).

We recall that a (p, q) -outer generalized inverse of a with prescribed idempotents p and q introduced by Djordjević and Wei in [8, Definition 2.1]. Given $p, q \in R^\bullet$, an element $a \in R$ has a (p, q) -outer generalized inverse $y \in R$, if

$$yay = y, \quad ya = p, \quad ay = 1 - q.$$

This (p, q) -outer inverse is unique if it exists, and we write $y = a_{p,q}^{(2)}$. If $a_{p,q}^{(2)}$ satisfies $aa_{p,q}^{(2)}a = a$, then $a_{p,q}^{(2)} = a_{p,q}^{(1,2)}$ is the (p, q) -reflexive generalized inverse of a . The set of all (p, q) -outer $((p, q)$ -reflexive) generalized invertible elements of R will be denoted by $R_{p,q}^{(2)}$ ($R_{p,q}^{(1,2)}$, resp.).

An element $a \in R$ is (von Neumann) regular if it has an inner inverse y , i.e. if there exists $y \in R$ such that $aya = a$. Any inner inverse of a will be denoted by a^- . The set of all regular elements of R will be denoted by R^- . An element $a \in R$ is called group invertible if there is $y \in R$ such that

$$aya = a, \quad yay = y, \quad ay = ya.$$

This y also is unique when it exists, and we denote it as $a^\#$. The set of all group invertible elements in R is denoted by $R^\#$.

Lemma 1.1. [3, Theorem 1] *Let $a, d \in R$. If $a \in R^\#$ and $da = ad$, then $a^\#d = da^\#$.*

For an element $a \in R$, we define the following image ideals

$$aR = \{ax : x \in R\}, \quad Ra = \{xa : x \in R\},$$

and kernel ideals

$$a^\circ = \{x \in R : ax = 0\}, \quad {}^\circ a = \{x \in R : xa = 0\}.$$

Let $a, y \in R$. Then $aR = yR$ if and only if there exist $u, v \in R$ such that $a = yu$ and $y = av$. Similarly, $Ra = Ry$ if and only if there exist $s, t \in R$ such that $a = sy$ and $y = ta$.

2 Reverse order laws for the (b, c) -inverse

It is well known that for nonsingular matrices A and B of the same size, we have

$$(AB)^{-1} = B^{-1}A^{-1}.$$

This equality is known as the reverse order law.

The study of the reverse order law for generalized inverses traces back to the work of Greville [11], who studied the reverse order law for the Moore-Penrose inverse of a matrix. Since then, a large amount of work has been devoted to the study of this problem, and equivalent conditions for the Moore-Penrose inverse reverse order law to hold have been proved in the setting of matrices, operators, or elements of rings with involution [10, 12, 21].

The “reverse order law” in a ring R , says that if $a, b \in R$ both invertible, then $(ab)^{-1} = b^{-1}a^{-1}$. It is a natural question whether this has extension to generalized inverses. The reverse order law for the generalized inverse is an useful computational tool in applications (solving linear equations in linear algebra or numerical analysis), and it is also interesting from the theoretical point of view.

In this section, we investigate the reverse order law for the (b, c) -inverse in rings. Precisely, we give necessary and sufficient conditions for rules $(aw)^{(b,c)} = w^{(b,s)}a^{(t,c)}$, $(aw)^{(b,c)} = (a^{(t,c)}aw)^{(b,s)}a^{(t,c)}$, $(aw)^{(b,c)} = w^{(b,s)}(aww^{(b,s)})^{(t,c)}$, $(a^{(t,c)}aw)^{(b,u)} = w^{(b,s)}a^{(t,c)}a$, $(aww^{(b,s)})^{(v,c)} = ww^{(b,s)}a^{(t,c)}$ and $(aw)^{(b,c)} = w^{(b,c)}(a^{(t,c)}aww^{(b,c)})^{(b,c)}a^{(t,c)}$ to be satisfied. As an application of these results, we obtain corresponding results related to the inverse along an element. Sufficient conditions for reverse order laws for the (p, q) -outer generalized inverse are presented. Also, we consider the more general case of reverse order law $(a_1a_2)^{(b_3,c_3)} = a_2^{(b_2,c_2)}a_1^{(b_1,c_1)}$.

The following lemmas will be useful in the sequel.

Lemma 2.1. *Let $a, b, c \in R$. Then the following statements are equivalent:*

- (i) $a^{(b,c)}$ exists;
- (ii) [4, Proposition 6.1] *there exists $y \in R$ such that $yay = y$, $yR = bR$ and $Ry = Rc$;*
- (iii) [17, Theorem 2.9] $b, c \in R^-$, and there exists $y \in R$ such that

$$y = bb^-y = yc^-c, \quad b = yab, \quad c = cay. \quad (1)$$

In this case, $y = a^{(b,c)}$.

Lemma 2.2. [5, Corollary 2.4, Remark 2.5] *Let $a, b, c, w \in R$. If $a^{(b,c)}$ exists, we have*

- (i) *if $ab = ba$ and $ca = ac$, then $a^{(b,c)}$ commutes with a ;*
- (ii) *if $ab = ba$ and $cb = bc$, then $a^{(b,c)}$ commutes with b ;*
- (iii) *if $ac = ca$ and $cb = bc$, then $a^{(b,c)}$ commutes with c .*

Specially, if $(b, c) = (d, d)$, a is invertible along d with (d, d) -inverse $a^{\parallel a}$ and $ad = da$, then $a^{\parallel d}$ commutes with a and d .

First, we will give the necessary and sufficient condition for the reverse order law

$$(aw)^{(b,c)} = w^{(b,s)}a^{(t,c)}.$$

Theorem 2.3. *Let $a, w, b, c, s, t \in R$ be such that $a^{(t,c)}$ and $w^{(b,s)}$ exist. Then the following statements are equivalent:*

- (i) $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = w^{(b,s)}a^{(t,c)}$;
- (ii) $b = w^{(b,s)}a^{(t,c)}awb$ and $c = caww^{(b,s)}a^{(t,c)}$.

Proof. Since $a^{(t,c)}$ exists, by Lemma 2.1, $c \in R^-$ and $a^{(t,c)}c^-c = a^{(t,c)}$. Similarly, $b \in R^-$ and $bb^-w^{(b,s)} = w^{(b,s)}$. Again using Lemma 2.1, $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = w^{(b,s)}a^{(t,c)}$ if and only if $b, c \in R^-$,

$$w^{(b,s)}a^{(t,c)} = bb^-w^{(b,s)}a^{(t,c)} = w^{(b,s)}a^{(t,c)}c^-c, \quad b = w^{(b,s)}a^{(t,c)}awb, \quad c = caww^{(b,s)}a^{(t,c)}.$$

Hence, (i) is equivalent to (ii). \square

If we suppose that a and w satisfy conditions $Ra \subseteq Rs$ and $wR \subseteq tR$, we get the next result as a consequence of Theorem 2.3.

Corollary 2.4. *Let $a, w, b, c, s, t \in R$ be such that $a^{(t,c)}$ and $w^{(b,s)}$ exist. If $Ra \subseteq Rs$ and $wR \subseteq tR$, then $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = w^{(b,s)}a^{(t,c)}$.*

Proof. Since $a^{(t,c)}$ and $w^{(b,s)}$ exist, by Lemma 2.1, $s, t \in R^-$. If $Ra \subseteq Rs$, then we get $a = xs$, for some $x \in R$, and so $a = xss^-s = as^-s$. Similarly, the condition $wR \subseteq tR$ and $t \in R^-$ imply $w = tt^-w$. Therefore,

$$\begin{aligned} w^{(b,s)}a^{(t,c)}awb &= w^{(b,s)}a^{(t,c)}a(tt^-w)b = w^{(b,s)}(a^{(t,c)}att^-)wb \\ &= w^{(b,s)}(tt^-)wb = w^{(b,s)}(tt^-w)b \\ &= w^{(b,s)}wb = b, \end{aligned}$$

and

$$\begin{aligned} caww^{(b,s)}a^{(t,c)} &= c(as^-s)ww^{(b,s)}a^{(t,c)} = ca(s^-sww^{(b,s)})a^{(t,c)} \\ &= cass^-a^{(t,c)} = caa^{(t,c)} = c. \end{aligned}$$

Applying Theorem 2.3, we can conclude that $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = w^{(b,s)}a^{(t,c)}$. \square

When $s = c$ and $t = b$, we have the following result.

Corollary 2.5. *Let $a, w, b, c \in R$ be such that $a^{(b,c)}$ and $w^{(b,c)}$ exist.*

- (i) *If $wb = bw$ and $ac = ca$, then $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = w^{(b,c)}a^{(b,c)}$.*
- (ii) *If $ab = ba$ and $ac = ca$, then $aw, wa \in R^{(b,c)}$, $(aw)^{(b,c)} = w^{(b,c)}a^{(b,c)}$ and $(wa)^{(b,c)} = a^{(b,c)}w^{(b,c)}$.*

Proof. (i) Since $wb = bw$ and $ac = ca$, we obtain

$$\begin{aligned} w^{(b,c)}a^{(b,c)}awb &= w^{(b,c)}a^{(b,c)}a(bw) = w^{(b,c)}(a^{(b,c)}ab)w = w^{(b,c)}bw = w^{(b,c)}wb = b, \\ caww^{(b,c)}a^{(b,c)} &= (ac)ww^{(b,c)}a^{(b,c)} = a(cww^{(b,c)})a^{(b,c)} = aca^{(b,c)} = caa^{(b,c)} = c. \end{aligned}$$

Therefore, by Theorem 2.3 (for $s = c$ and $t = b$), $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = w^{(b,c)}a^{(b,c)}$.

(ii) First, we will prove that $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = w^{(b,c)}a^{(b,c)}$. By the proof of (i), the condition $ac = ca$ gives $c = caww^{(b,c)}a^{(b,c)}$. Note that from Lemma 2.2, if $ab = ba$ and $ac = ca$, then $a^{(b,c)}a = aa^{(b,c)}$. Next we will prove that $w^{(b,c)}a^{(b,c)}awb = b$. Indeed,

$$\begin{aligned} w^{(b,c)}a^{(b,c)}awb &= (w^{(b,c)}c^-c)a^{(b,c)}awb = w^{(b,c)}c^-c(aa^{(b,c)})wb \\ &= w^{(b,c)}c^-(caa^{(b,c)})wb = w^{(b,c)}c^-cwb \\ &= w^{(b,c)}wb = b. \end{aligned}$$

Similarly, we can prove that $wa \in R^{(b,c)}$ and $(wa)^{(b,c)} = a^{(b,c)}w^{(b,c)}$. □

If $b = c = d$, we have the following reverse order law for the Mary inverse.

Corollary 2.6. [26, Theorem 2.14] *Let $a, w, d \in R$ be such that $a^{\parallel d}$ and $w^{\parallel d}$ exist. If $ad = da$, then $aw, wa \in R^{\parallel d}$, $(aw)^{\parallel d} = w^{\parallel d}a^{\parallel d}$ and $(wa)^{\parallel d} = a^{\parallel d}w^{\parallel d}$.*

Next, we consider the following reverse order law

$$(aw)^{(b,c)} = (a^{(t,c)}aw)^{(b,s)}a^{(t,c)}.$$

Theorem 2.7. *Let $a, w, b, c, s, t \in R$ be such that $a^{(t,c)}$ and $(a^{(t,c)}aw)^{(b,s)}$ exist. Then the following statements are equivalent:*

- (i) $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = (a^{(t,c)}aw)^{(b,s)}a^{(t,c)}$;
- (ii) $c = caw(a^{(t,c)}aw)^{(b,s)}a^{(t,c)}$.

Proof. Using Lemma 2.1, $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = (a^{(t,c)}aw)^{(b,s)}a^{(t,c)}$ hold if and only if

$$b, c \in R^-, \quad (a^{(t,c)}aw)^{(b,s)}a^{(t,c)} = bb^-(a^{(t,c)}aw)^{(b,s)}a^{(t,c)} = (a^{(t,c)}aw)^{(b,s)}a^{(t,c)}c^-c,$$

and

$$b = (a^{(t,c)}aw)^{(b,s)}a^{(t,c)}awb, \quad c = caw(a^{(t,c)}aw)^{(b,s)}a^{(t,c)}.$$

Since $a^{(t,c)}$ and $(a^{(t,c)}aw)^{(b,s)}$ exist, again using Lemma 2.1, we have $b, c \in R^-$,

$$(a^{(t,c)}aw)^{(b,s)} = bb^-(a^{(t,c)}aw)^{(b,s)}, \quad a^{(t,c)} = a^{(t,c)}c^-c, \quad b = (a^{(t,c)}aw)^{(b,s)}a^{(t,c)}awb.$$

Therefore, (i) is equivalent to (ii). □

Corollary 2.8. *Let $a, w, b, c, s, t \in R$ be such that $a^{(t,c)}$ and $(a^{(t,c)}aw)^{(b,s)}$ exist.*

- (i) *If $Ra \subseteq Rs$, then $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = (a^{(t,c)}aw)^{(b,s)}a^{(t,c)}$.*

(ii) If $aw \in R^{(b,c)}$ and $(a^{(t,c)}aw)^{(b,s)} = (aw)^{(b,c)}a$, then $(aw)^{(b,c)} = (a^{(t,c)}aw)^{(b,s)}a^{(t,c)}$.

Proof. (i) Similar discuss as Corollary 2.4, we get $a = as^{-1}$. As $a^{(t,c)}$ and $(a^{(t,c)}aw)^{(b,s)}$ exist, we know that $caa^{(t,c)} = c$ and $sa^{(t,c)}aw(a^{(t,c)}aw)^{(b,s)} = s$. Therefore,

$$\begin{aligned} caw(a^{(t,c)}aw)^{(b,s)}a^{(t,c)} &= (caa^{(t,c)})aw(a^{(t,c)}aw)^{(b,s)}a^{(t,c)} \\ &= c(as^{-1})a^{(t,c)}aw(a^{(t,c)}aw)^{(b,s)}a^{(t,c)} \\ &= cas^{-1}(sa^{(t,c)}aw(a^{(t,c)}aw)^{(b,s)})a^{(t,c)} \\ &= cas^{-1}sa^{(t,c)} = caa^{(t,c)} = c. \end{aligned}$$

Using Theorem 2.7, $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = (a^{(t,c)}aw)^{(b,s)}a^{(t,c)}$.

(ii) Since $a^{(t,c)}$ and $(aw)^{(b,c)}$ exist, we get $caa^{(t,c)} = c$ and $caw(aw)^{(b,c)} = c$. So the hypothesis $(a^{(t,c)}aw)^{(b,s)} = (aw)^{(b,c)}a$ gives

$$caw(a^{(t,c)}aw)^{(b,s)}a^{(t,c)} = caw((aw)^{(b,c)}a)a^{(t,c)} = (caw(aw)^{(b,c)})aa^{(t,c)} = caa^{(t,c)} = c.$$

Hence, by Theorem 2.7, $(aw)^{(b,c)} = (a^{(t,c)}aw)^{(b,s)}a^{(t,c)}$. □

In a similar way we can prove the related results for $(aw)^{(b,c)} = w^{(b,s)}(aww^{(b,s)})^{(t,c)}$.

Theorem 2.9. Let $a, w, b, c, s, t \in R$ be such that $(aww^{(b,s)})^{(t,c)}$ and $w^{(b,s)}$ exist. Then the following are equivalent:

(i) $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = w^{(b,s)}(aww^{(b,s)})^{(t,c)}$;

(ii) $b = w^{(b,s)}(aww^{(b,s)})^{(t,c)}awb$.

Corollary 2.10. Let $a, w, b, c, s, t \in R$ be such that $(aww^{(b,s)})^{(t,c)}$ and $w^{(b,s)}$ exist.

(i) If $wR \subseteq tR$, then $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = w^{(b,s)}(aww^{(b,s)})^{(t,c)}$.

(ii) If $aw \in R^{(b,c)}$ and $(aww^{(b,s)})^{(t,c)} = w(aw)^{(b,c)}$, then $(aw)^{(b,c)} = w^{(b,s)}(aww^{(b,s)})^{(t,c)}$.

Also, if $s = c$ and $t = b$, we get the following result.

Corollary 2.11. Let $a, w, b, c \in R$.

(i) Suppose that $a^{(b,c)}$ and $(a^{(b,c)}aw)^{(b,c)}$ exist. If $ac = ca$, then $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = (a^{(b,c)}aw)^{(b,c)}a^{(b,c)}$.

(ii) Suppose that $(aww^{(b,c)})^{(b,c)}$ and $w^{(b,c)}$ exist. If $wb = bw$, then $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = w^{(b,c)}(aww^{(b,c)})^{(b,c)}$.

Proof. (i) Since $a^{(b,c)}$ and $(a^{(b,c)}aw)^{(b,c)}$ exist, we get $caa^{(b,c)} = c = c(a^{(b,c)}aw)(a^{(b,c)}aw)^{(b,c)}$. If $ac = ca$, then

$$\begin{aligned} caw(a^{(b,c)}aw)^{(b,c)}a^{(b,c)} &= (caa^{(b,c)})aw(a^{(b,c)}aw)^{(b,c)}a^{(b,c)} \\ &= aca^{(b,c)}aw(a^{(b,c)}aw)^{(b,c)}a^{(b,c)} \\ &= aca^{(b,c)} = caa^{(b,c)} = c. \end{aligned}$$

By Theorem 2.7 (for $s = c$ and $t = b$), $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = (a^{(b,c)}aw)^{(b,c)}a^{(b,c)}$.

(ii) Similarly as (i). □

Corollary 2.12. *Let $a, w, b, c \in R$.*

- (i) *Suppose that $a^{\parallel d}$ and $(a^{\parallel d}aw)^{\parallel d}$ exist. If $ad = da$, then $aw \in R^{\parallel d}$ and $(aw)^{\parallel d} = (a^{\parallel d}aw)^{\parallel d}a^{\parallel d}$.*
- (ii) *Suppose that $(aww^{\parallel d})^{\parallel d}$ and $w^{\parallel d}$ exist. If $wd = dw$, then $aw \in R^{\parallel d}$ and $(aw)^{\parallel d} = w^{\parallel d}(aww^{\parallel d})^{\parallel d}$.*

Now the following reverse order law is studied

$$(a^{(t,c)}aw)^{(b,u)} = w^{(b,s)}a^{(t,c)}a.$$

Theorem 2.13. *Let $a, w, b, c, s, t, u \in R$ be such that $a^{(t,c)}$ and $w^{(b,s)}$ exist. Then the following statements are equivalent:*

- (i) $a^{(t,c)}aw \in R^{(b,u)}$ and $(a^{(t,c)}aw)^{(b,u)} = w^{(b,s)}a^{(t,c)}a$;
- (ii) $u \in R^-$, $w^{(b,s)}a^{(t,c)}a(1 - uu^-) = 0$, $b = w^{(b,s)}a^{(t,c)}awb$ and $c = ca^{(t,c)}aww^{(b,s)}a^{(t,c)}a$.

Proof. Using Lemma 2.1, (i) holds if and only if $b, u \in R^-$,

$$w^{(b,s)}a^{(t,c)}a = bb^-w^{(b,s)}a^{(t,c)}a = w^{(b,s)}a^{(t,c)}auu^-,$$

$$b = w^{(b,s)}a^{(t,c)}aa^{(t,c)}awb, \quad \text{and} \quad c = ca^{(t,c)}aww^{(b,s)}a^{(t,c)}a.$$

Since $a^{(t,c)}$ and $w^{(b,s)}$ exist, it follows that $b \in R^-$, $bb^-w^{(b,s)} = w^{(b,s)}$ and $a^{(t,c)}aa^{(t,c)} = a^{(t,c)}$. Therefore, (i) is equivalent to (ii). \square

Using Theorem 2.13, we get some sufficient conditions for $(aw)^{(b,c)} = w^{(b,s)}a^{(t,c)}$.

Corollary 2.14. *Let $a, w, b, c, s, t, u \in R$ be such that $a^{(t,c)}$, $w^{(b,s)}$, $(aw)^{(b,c)}$ and $(a^{(t,c)}aw)^{(b,u)}$ exist. If $(aw)^{(b,c)} = (a^{(t,c)}aw)^{(b,u)}a^{(t,c)}$ and one of the following equivalent statements holds:*

- (i) $(a^{(t,c)}aw)^{(b,u)} = w^{(b,s)}a^{(t,c)}a$;
- (ii) $w^{(b,s)}a^{(t,c)}a(1 - uu^-) = 0$, $b = w^{(b,s)}a^{(t,c)}awb$ and $c = ca^{(t,c)}aww^{(b,s)}a^{(t,c)}a$,

then $(aw)^{(b,c)} = w^{(b,s)}a^{(t,c)}$.

Proof. Since $(a^{(t,c)}aw)^{(b,u)}$ exists, we have $u \in R^-$, so (i) is equivalent to (ii) by Theorem 2.13. Therefore,

$$(aw)^{(b,c)} = (a^{(t,c)}aw)^{(b,u)}a^{(t,c)} = w^{(b,s)}a^{(t,c)}aa^{(t,c)} = w^{(b,s)}a^{(t,c)}.$$

\square

Note that if $u = s$ in Corollary 2.14, using Theorem 2.7, then we can replace the condition $(aw)^{(b,c)} = (a^{(t,c)}aw)^{(b,u)}a^{(t,c)}$ by $c = caw(a^{(t,c)}aw)^{(b,s)}a^{(t,c)}$ to obtain another sufficient condition for $(aw)^{(b,c)} = w^{(b,s)}a^{(t,c)}$. Here we left it to reader.

Analogously, we have the following results.

Theorem 2.15. Let $a, w, b, c, s, t, v \in R$ be such that $a^{(t,c)}$ and $w^{(b,s)}$ exist. Then the following statements are equivalent:

- (i) $aww^{(b,s)} \in R^{(v,c)}$ and $(aww^{(b,s)})^{(v,c)} = ww^{(b,s)}a^{(t,c)}$;
- (ii) $v \in R^-$, $(1 - vv^-)ww^{(b,s)}a^{(t,c)} = 0$, $b = ww^{(b,s)}a^{(t,c)}aww^{(b,s)}b$ and $c = caww^{(b,s)}a^{(t,c)}$.

Corollary 2.16. Let $a, w, b, c, s, t, v \in R$ be such that $a^{(t,c)}$, $w^{(b,s)}$, $(aw)^{(b,c)}$ and $(aww^{(b,s)})^{(v,c)}$ exist. If $(aw)^{(b,c)} = w^{(b,s)}(aww^{(b,s)})^{(t,c)}$ and one of the following equivalent statements holds:

- (i) $aww^{(b,s)} \in R^{(v,c)}$ and $(aww^{(b,s)})^{(v,c)} = ww^{(b,s)}a^{(t,c)}$;
- (ii) $(1 - vv^-)ww^{(b,s)}a^{(t,c)} = 0$, $b = ww^{(b,s)}a^{(t,c)}aww^{(b,s)}b$ and $c = caww^{(b,s)}a^{(t,c)}$,

then $(aw)^{(b,c)} = w^{(b,s)}a^{(t,c)}$.

Corollary 2.17. Let $a, w, b, c, u, v \in R$ be such that $a^{(b,c)}$ and $w^{(b,c)}$ exist. Then

- (i) If $wb = bw$, then $a^{(b,c)}aw \in R^{(b,u)}$ and $(a^{(b,c)}aw)^{(b,u)} = w^{(b,c)}a^{(b,c)}a$ if and only if

$$u \in R^-, \quad w^{(b,c)}a^{(b,c)}a(1 - uu^-) = 0, \quad c = ca^{(b,c)}a.$$

- (ii) If $ca = ac$, then $aww^{(b,c)} \in R^{(v,c)}$ and $(aww^{(b,c)})^{(v,c)} = ww^{(b,c)}a^{(b,c)}$ if and only if

$$v \in R^-, \quad (1 - vv^-)ww^{(b,c)}a^{(b,c)} = 0, \quad b = ww^{(b,c)}b.$$

Proof. (i) It follows by Theorem 2.13 and

$$\begin{aligned} ca^{(b,c)}aww^{(b,c)}a^{(b,c)}a &= ca^{(b,c)}a(wb)b^-w^{(b,c)}a^{(b,c)}a = c(a^{(b,c)}ab)wb^-w^{(b,c)}a^{(b,c)}a \\ &= c(bw)b^-w^{(b,c)}a^{(b,c)}a = cw(bb^-w^{(b,c)})a^{(b,c)}a \\ &= cww^{(b,c)}a^{(b,c)}a = ca^{(b,c)}a. \end{aligned}$$

- (ii) In the similar way as (i). □

Next, we investigate the following reverse order law

$$(aw)^{(b,c)} = w^{(b,c)}(a^{(t,c)}aww^{(b,c)})^{(b,c)}a^{(t,c)}.$$

Theorem 2.18. Let $a, w, b, c, t \in R$ be such that $a^{(t,c)}$, $w^{(b,c)}$ and $(a^{(t,c)}aww^{(b,c)})^{(b,c)}$ exist. Then the following statements are equivalent:

- (i) $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = w^{(b,c)}(a^{(t,c)}aww^{(b,c)})^{(b,c)}a^{(t,c)}$;
- (ii) $a^{(t,c)}aw \in R^{(b,c)}$, $aww^{(b,c)} \in R^{(t,c)}$, $(a^{(t,c)}aw)^{(b,c)} = w^{(b,c)}(a^{(t,c)}aww^{(b,c)})^{(b,c)}$ and $(aww^{(b,c)})^{(t,c)} = (a^{(t,c)}aww^{(b,c)})^{(b,c)}a^{(t,c)}$.

Proof. (i) \Rightarrow (ii): If (i) holds, first we will observe that

$$a^{(t,c)}aw \in R^{(b,c)} \quad \text{and} \quad (a^{(t,c)}aw)^{(b,c)} = w^{(b,c)}(a^{(t,c)}aww^{(b,c)})^{(b,c)}. \quad (2)$$

In fact, since $a^{(t,c)}$, $w^{(b,c)}$ and $(a^{(t,c)}aww^{(b,c)})^{(b,c)}$ exist, by Lemma 2.1, we know $b, c, t \in R^-$,

$$bb^-w^{(b,c)} = w^{(b,c)}, \quad (a^{(t,c)}aww^{(b,c)})^{(b,c)} = (a^{(t,c)}aww^{(b,c)})^{(b,c)}c^-c,$$

$$ca^{(t,c)}aww^{(b,c)}(a^{(t,c)}aww^{(b,c)})^{(b,c)} = c.$$

Then $w^{(b,c)}(a^{(t,c)}aww^{(b,c)})^{(b,c)} = bb^-w^{(b,c)}(a^{(t,c)}aww^{(b,c)})^{(b,c)} = w^{(b,c)}(a^{(t,c)}aww^{(b,c)})^{(b,c)}c^-c$. As (i) holds, we have

$$w^{(b,c)}(a^{(t,c)}aww^{(b,c)})^{(b,c)}a^{(t,c)}awb = (aw)^{(b,c)}(aw)b = b.$$

Again using Lemma 2.1, we can conclude that (2) holds.

Similarly, we can obtain that $aww^{(b,c)} \in R^{(t,c)}$ and $(aww^{(b,c)})^{(t,c)} = (a^{(t,c)}aww^{(b,c)})^{(b,c)}a^{(t,c)}$.

(ii) \Rightarrow (i): If (ii) holds, by Lemma 2.1, we see $b, c \in R^-$,

$$w^{(b,c)}(a^{(t,c)}aww^{(b,c)})^{(b,c)} = bb^-w^{(b,c)}(a^{(t,c)}aww^{(b,c)})^{(b,c)},$$

$$(a^{(t,c)}aww^{(b,c)})^{(b,c)}a^{(t,c)} = (a^{(t,c)}aww^{(b,c)})^{(b,c)}a^{(t,c)}c^-c,$$

$$w^{(b,c)}(a^{(t,c)}aww^{(b,c)})^{(b,c)}a^{(t,c)}awb = (a^{(t,c)}aw)^{(b,c)}a^{(t,c)}awb = b,$$

$$caww^{(b,c)}(a^{(t,c)}aww^{(b,c)})^{(b,c)}a^{(t,c)} = caww^{(b,c)}(aww^{(b,c)})^{(t,c)} = c.$$

Therefore, we can conclude that (i) holds. \square

Remark 2.19. Since the Mary inverse, Bott-Duffin (e, f) -inverse, image-kernel (p, q) -inverse are all the special cases of (b, c) -inverse if we choose b and c appropriately, related results for these inverses are obtained.

The reverse order law for the (p, q) -outer generalized inverse are considered in the following sequel.

Proposition 2.20. Let $p, q, r, s \in R^\bullet$ and let $a, b \in R$ be such that $a_{p,q}^{(2)}$ and $b_{s,1-p}^{(2)}$ exist. Then:

$$(i) \quad ab \in R_{s,q}^{(2)} \quad \text{and} \quad (ab)_{s,q}^{(2)} = b_{s,1-p}^{(2)}a_{p,q}^{(2)};$$

$$(ii) \quad a_{p,q}^{(2)}ab \in R_{s,1-p}^{(2)} \quad \text{and} \quad (a_{p,q}^{(2)}ab)_{s,1-p}^{(2)} = b_{s,1-p}^{(2)}a_{p,q}^{(2)}a;$$

$$(iii) \quad abb_{s,1-p}^{(2)} \in R_{p,q}^{(2)} \quad \text{and} \quad (abb_{s,1-p}^{(2)})_{p,q}^{(2)} = bb_{s,1-p}^{(2)}a_{p,q}^{(2)}.$$

Proposition 2.21. Let $p, q, r, s, t \in R^\bullet$ and $a, b \in R$.

$$(i) \quad \text{If } a \in R_{p,q}^{(2)} \text{ and } ab \in R_{t,q}^{(2)}, \text{ then } a_{p,q}^{(2)}ab \in R_{t,1-p}^{(2)} \text{ and } (a_{p,q}^{(2)}ab)_{t,1-p}^{(2)} = (ab)_{t,q}^{(2)}a.$$

- (ii) If $b \in R_{s,t}^{(2)}$ and $ab \in R_{s,r}^{(2)}$, then $abb_{s,t}^{(2)} \in R_{1-t,r}^{(2)}$ and $(abb_{s,t}^{(2)})_{1-t,r}^{(2)} = b(ab)_{s,r}^{(2)}$.
- (iii) If $a \in R_{p,q}^{(2)}$, $b \in R_{s,t}^{(2)}$ and $ab \in R_{s,q}^{(2)}$, then $a_{p,q}^{(2)}abb_{s,t}^{(2)} \in R_{1-t,1-p}^{(2)}$ and $(a_{p,q}^{(2)}abb_{s,t}^{(2)})_{1-t,1-p}^{(2)} = b(ab)_{s,q}^{(2)}a$.

Proposition 2.22. Let $p, q, r, s, t \in R^\bullet$ and $a, b \in R$.

- (i) If $a \in R_{p,q}^{(1,2)}$ and $a_{p,q}^{(1,2)}ab \in R_{t,1-p}^{(2)}$, then $ab \in R_{t,q}^{(2)}$ and $(ab)_{t,q}^{(2)} = (a_{p,q}^{(1,2)}ab)_{t,1-p}^{(2)}a_{p,q}^{(1,2)}$.
- (ii) If $b \in R_{s,t}^{(1,2)}$ and $abb_{s,t}^{(1,2)} \in R_{1-t,r}^{(2)}$, then $ab \in R_{s,r}^{(2)}$ and $(ab)_{s,r}^{(2)} = b_{s,t}^{(1,2)}(abb_{s,t}^{(1,2)})_{1-t,r}^{(2)}$.
- (iii) If $a \in R_{p,q}^{(1,2)}$, $b \in R_{s,t}^{(1,2)}$ and $a_{p,q}^{(1,2)}abb_{s,t}^{(1,2)} \in R_{1-t,1-p}^{(2)}$, then $ab \in R_{s,q}^{(2)}$ and $(ab)_{s,q}^{(2)} = b_{s,t}^{(1,2)}(a_{p,q}^{(1,2)}abb_{s,t}^{(1,2)})_{1-t,1-p}^{(2)}a_{p,q}^{(1,2)}$.

Finally, we consider the more general case of reverse order law $(a_1a_2)^{(b_3,c_3)} = a_2^{(b_2,c_2)}a_1^{(b_1,c_1)}$ when a_i is (b_i, c_i) -invertible with (b_i, c_i) -inverse $a_i^{(b_i, c_i)}$ ($i = 1, 2$). Before to discuss it, we need the following lemma which can be seen in [17]. We will give the proof of the following result for the sake of completeness.

Lemma 2.23. [17, Theorem 2.11] Let $a, b, c \in R$ be such that $a^{(b,c)}$ exists. Let $w \in R$ be such that $wR = bR$ and $w^\circ = c^\circ$ (or $Rw = Rc$). Then $aw, wa \in R^\#$, $a^{(b,c)} = w(aw)^\# = (wa)^\#w$, and $w = a^{(b,c)}aw = waa^{(b,c)}$.

Proof. If $a^{(b,c)}$ exists, by Lemma 2.1, $b, c \in R^-$. The condition $b \in R^-$ and $wR = bR$ imply that $w \in R^-$. So $w^\circ = c^\circ$ is equivalent to $Rw = Rc$.

By Lemma 2.1, we have $a^{(b,c)}aa^{(b,c)} = a^{(b,c)}$, $a^{(b,c)}R = bR = wR$ and $Ra^{(b,c)} = Rc = Rw$. Then there exist $x, y, z \in R$ such that $w = a^{(b,c)}x = ya^{(b,c)}$ and $a^{(b,c)} = wz$. So $w = a^{(b,c)}aa^{(b,c)}x = a^{(b,c)}aw$. Similarly, $w = waa^{(b,c)}$.

Let $u \in (aw)^\circ$. We have $awu = 0$, which gives $wu = a^{(b,c)}awu = 0$, i.e., $u \in w^\circ$. Hence $(aw)^\circ = w^\circ$. Since $a^{(b,c)}$ exists, using [4, Proposition 2.7], $R = abR \oplus c^\circ$. Thus $R = awR \oplus (aw)^\circ$. From the proof of [13, Proposition 7, p.205], $aw \in R^\#$.

Let $v = w(aw)^\#$. We can show that v is the (b, c) -inverse of a . Indeed, as $bR = wR$ and $Rw = Rc$, there are $x_1, x_2, y_1, y_2 \in R$ such that $b = wx_1$, $w = by_1 = x_2c$ and $c = y_2w$. Thus,

$$bb^-v = bb^-w(aw)^\# = bb^-by_1(aw)^\# = by_1(aw)^\# = w(aw)^\# = v.$$

Similarly, we get $vc^-c = v$. Since $aw((aw)^\#aw - 1) = 0$ and $(aw)^\circ = w^\circ$, we have $w = w(aw)^\#aw$. Hence,

$$vab = w(aw)^\#ab = w(aw)^\#awx_1 = wx_1 = b,$$

$$cav = caw(aw)^\# = y_2waw(aw)^\# = y_2w = c.$$

Therefore, by Lemma 2.1, $v = a^{(b,c)}$.

Similarly, we can prove that $wa \in R^\#$ and $a^{(b,c)} = (wa)^\#w$. □

Theorem 2.24. Let $a_i, b_i, c_i, b_3, c_3 \in R$ ($i = 1, 2$) be such that a_i is (b_i, c_i) -invertible with (b_i, c_i) -inverse $a_i^{(b_i, c_i)}$ ($i = 1, 2$). Let $a'_1, a'_2 \in R$ satisfy

$$a'_i R = b_i R, \quad Ra'_i = Rc_i \quad (i = 1, 2),$$

$$a'_2 a'_1 R = b_3 R, \quad Ra'_2 a'_1 = Rc_3.$$

If $a_1^{(b_1, c_1)} a_1$ commutes with $a_2 a'_2$ and $a_2 a_2^{(b_2, c_2)}$ commutes with $a'_1 a_1$, then $a_1 a_2$ is (b_3, c_3) -invertible and

$$(a_1 a_2)^{(b_3, c_3)} = a_2^{(b_2, c_2)} a_1^{(b_1, c_1)}.$$

Proof. First, if a_i is (b_i, c_i) -invertible with (b_i, c_i) -inverse $a_i^{(b_i, c_i)}$ ($i = 1, 2$), by Lemma 2.1, it follows that

$$a_i^{(b_i, c_i)} a_i a_i^{(b_i, c_i)} = a_i^{(b_i, c_i)}, \quad a_i^{(b_i, c_i)} R = b_i R, \quad Ra_i^{(b_i, c_i)} = Rc_i \quad (i = 1, 2). \quad (3)$$

Since $a_i^{(b_i, c_i)}$ ($i = 1, 2$) exist and $a'_i R = b_i R$, $Ra'_i = Rc_i$ ($i = 1, 2$), as an application of Lemma 2.23, we have $a_i a'_i, a'_i a_i$ ($i = 1, 2$) $\in R^\#$, and

$$a_i^{(b_i, c_i)} = a'_i (a_i a'_i)^\# = (a'_i a_i)^\# a'_i, \quad \text{and} \quad a'_i = a_i^{(b_i, c_i)} a_i a'_i = a'_i a_i a_i^{(b_i, c_i)} \quad (i = 1, 2). \quad (4)$$

As $a'_1 a_1, a_2 a'_2 \in R^\#$, $a_1^{(b_1, c_1)} a_1$ commutes with $a_2 a'_2$ and $a_2 a_2^{(b_2, c_2)}$ commutes with $a'_1 a_1$, by Lemma 1.1, we obtain the following equations

$$a_1^{(b_1, c_1)} a_1 (a_2 a'_2)^\# = (a_2 a'_2)^\# a_1^{(b_1, c_1)} a_1, \quad (5)$$

$$a_2 a_2^{(b_2, c_2)} (a'_1 a_1)^\# = (a'_1 a_1)^\# a_2 a_2^{(b_2, c_2)}. \quad (6)$$

Therefore, using equations (4) and (5) we have

$$\begin{aligned} a_1^{(b_1, c_1)} a_1 a_2 a_2^{(b_2, c_2)} &\stackrel{(4)}{=} a_1^{(b_1, c_1)} a_1 a_2 (a'_2 (a_2 a'_2)^\#) = (a_1^{(b_1, c_1)} a_1 a_2 a'_2) (a_2 a'_2)^\# \\ &= (a_2 a'_2 a_1^{(b_1, c_1)} a_1) (a_2 a'_2)^\# = a_2 a'_2 (a_1^{(b_1, c_1)} a_1 (a_2 a'_2)^\#) \\ &\stackrel{(5)}{=} a_2 a'_2 ((a_2 a'_2)^\# a_1^{(b_1, c_1)} a_1) \stackrel{(4)}{=} a_2 a_2^{(b_2, c_2)} a_1^{(b_1, c_1)} a_1. \end{aligned} \quad (7)$$

Now we will prove that $a_1 a_2 \in R^{(b_3, c_3)}$ and $(a_1 a_2)^{(b_3, c_3)} = a_2^{(b_2, c_2)} a_1^{(b_1, c_1)}$. If we prove that

$$(a_2^{(b_2, c_2)} a_1^{(b_1, c_1)}) (a_1 a_2) (a_2^{(b_2, c_2)} a_1^{(b_1, c_1)}) = a_2^{(b_2, c_2)} a_1^{(b_1, c_1)}, \quad (8)$$

$$a_2^{(b_2, c_2)} a_1^{(b_1, c_1)} R = b_3 R, \quad Ra_2^{(b_2, c_2)} a_1^{(b_1, c_1)} = Rc_3. \quad (9)$$

Then by Lemma 2.1, the equation $(a_1 a_2)^{(b_3, c_3)} = a_2^{(b_2, c_2)} a_1^{(b_1, c_1)}$ hold. Indeed,

$$\begin{aligned} (a_2^{(b_2, c_2)} a_1^{(b_1, c_1)}) (a_1 a_2) (a_2^{(b_2, c_2)} a_1^{(b_1, c_1)}) &= a_2^{(b_2, c_2)} (a_1^{(b_1, c_1)} a_1 a_2 a_2^{(b_2, c_2)}) a_1^{(b_1, c_1)} \\ &\stackrel{(7)}{=} a_2^{(b_2, c_2)} (a_2 a_2^{(b_2, c_2)} a_1^{(b_1, c_1)} a_1) a_1^{(b_1, c_1)} \\ &= (a_2^{(b_2, c_2)} a_2 a_2^{(b_2, c_2)}) (a_1^{(b_1, c_1)} a_1 a_1^{(b_1, c_1)}) \\ &\stackrel{(3)}{=} a_2^{(b_2, c_2)} a_1^{(b_1, c_1)}, \end{aligned}$$

that is, the equation (8) holds.

Since $a_i^{(b_i, c_i)} a_i a_i^{(b_i, c_i)} = a_i^{(b_i, c_i)}$, we have $a_i^{(b_i, c_i)} R = a_i^{(b_i, c_i)} a_i R$, $R a_i a_i^{(b_i, c_i)} = R a_i^{(b_i, c_i)}$ ($i = 1, 2$). Thus,

$$\begin{aligned}
a_2^{(b_2, c_2)} a_1^{(b_1, c_1)} R &= a_2^{(b_2, c_2)} a_1^{(b_1, c_1)} a_1 R \stackrel{(4)}{=} (a_2' (a_2 a_2')^\#) a_1^{(b_1, c_1)} a_1 R = a_2' ((a_2 a_2')^\# a_1^{(b_1, c_1)} a_1) R \\
&\stackrel{(5)}{=} a_2' (a_1^{(b_1, c_1)} a_1 (a_2 a_2')^\#) R \stackrel{(4)}{=} a_2' (a_1' (a_1 a_1')^\#) a_1 (a_2 a_2')^\# R \\
&\subseteq a_2' a_1' R = a_2' b_1 R \stackrel{(3)}{=} a_2' a_1^{(b_1, c_1)} R = a_2' a_1^{(b_1, c_1)} a_1 R \\
&\stackrel{(4)}{=} (a_2^{(b_2, c_2)} a_2 a_2') a_1^{(b_1, c_1)} a_1 R = a_2^{(b_2, c_2)} (a_2 a_2' a_1^{(b_1, c_1)} a_1) R \\
&= a_2^{(b_2, c_2)} (a_1^{(b_1, c_1)} a_1 a_2 a_2') R \\
&\subseteq a_2^{(b_2, c_2)} a_1^{(b_1, c_1)} R.
\end{aligned}$$

Consequently, $a_2^{(b_2, c_2)} a_1^{(b_1, c_1)} R = a_2' a_1' R = b_3 R$, i.e. the left equation of (9) holds.

Similarly, we get

$$\begin{aligned}
R a_2^{(b_2, c_2)} a_1^{(b_1, c_1)} &= R a_2 a_2^{(b_2, c_2)} a_1^{(b_1, c_1)} \stackrel{(4)}{=} R a_2 a_2^{(b_2, c_2)} ((a_1' a_1)^\# a_1') = R (a_2 a_2^{(b_2, c_2)} (a_1' a_1)^\#) a_1' \\
&\stackrel{(6)}{=} R ((a_1' a_1)^\# a_2 a_2^{(b_2, c_2)}) a_1' \stackrel{(4)}{=} R (a_1' a_1)^\# a_2 ((a_2' a_2)^\# a_2') a_1' \\
&\subseteq R a_2' a_1' = R c_2 a_1' \stackrel{(3)}{=} R a_2^{(b_2, c_2)} a_1' = R a_2 a_2^{(b_2, c_2)} a_1' \\
&\stackrel{(4)}{=} R a_2 a_2^{(b_2, c_2)} (a_1' a_1 a_1^{(b_1, c_1)}) = R (a_2 a_2^{(b_2, c_2)} a_1' a_1) a_1^{(b_1, c_1)} \\
&= R (a_1' a_1 a_2 a_2^{(b_2, c_2)}) a_1^{(b_1, c_1)} \\
&\subseteq R a_2^{(b_2, c_2)} a_1^{(b_1, c_1)}.
\end{aligned}$$

Thus, $R a_2^{(b_2, c_2)} a_1^{(b_1, c_1)} = R a_2' a_1' = R c_3$, i.e. the right equation of (9) holds. \square

Let $b_i = c_i = d_i$ ($i = 1, 2, 3$) in Theorem 2.24, then we have the following result for Mary inverse.

Corollary 2.25. *Let $a_i, d_i \in R$ ($i = 1, 2$) be such that $a_i^{\parallel d_i}$ ($i = 1, 2$) exists. If there exists $d_3 \in R$ such that $d_2 d_1 R = d_3 R$, $R d_2 d_1 = R d_3$, and $a_1^{\parallel d_1} a_1$ commutes with $a_2 d_2$ and $a_2 a_2^{\parallel d_2}$ commutes with $d_1 a_1$, then $a_1 a_2 \in R^{\parallel d_3}$ and*

$$(a_1 a_2)^{\parallel d_3} = a_2^{\parallel d_2} a_1^{\parallel d_1}.$$

Proof. By [18, Theorem 7], we know that if $a^{\parallel d}$ exists, then $ad, da \in R^\#$ and $a^{\parallel d} = d(ad)^\# = (da)^\#$. Moreover, by the definition of Mary inverse, we obtain that $d = a^{\parallel d} a d = d a a^{\parallel d}$, $a^{\parallel d} R = d R$ and $R a^{\parallel d} = R d$. Take $a_i' = d_i$ ($i = 1, 2$) and $b_i = c_i = d_i$ ($i = 1, 2, 3$) in Theorem 2.24, as required. \square

As an application of this theorem, we first recall the basic properties of outer generalized inverses with prescribed range and kernel (see [9]). Let X and Y be Banach spaces and let $\mathcal{L}(X, Y)$ denote the set of all bounded operators from X to Y . For $A \in \mathcal{L}(X, Y)$ we use $N(A)$ and $R(A)$ to denote the range and the kernel of A . We say that $C \in \mathcal{L}(Y, X)$ is an outer

generalized inverse of A , if $CAC = C$. Let T and S be subspaces of X and Y , resp., such that there exists an outer generalized inverse $A_{T,S}^{(2)} \in \mathcal{L}(Y, X)$ of A with range equal to T and kernel equal to S , i.e., $A_{T,S}^{(2)}$ satisfies

$$A_{T,S}^{(2)}AA_{T,S}^{(2)} = A_{T,S}^{(2)}, \quad R(A_{T,S}^{(2)}) = T, \quad N(A_{T,S}^{(2)}) = S.$$

If A, T and S given as above, then $A_{T,S}^{(2)}$ exists if and only if T and S , respectively, are closed and complemented subspaces of X and Y , the reduction $A_T : T \rightarrow A(T)$ is invertible and $A(T) \oplus S = Y$. In this case $A_{T,S}^{(2)}$ is unique.

The following result can be seen in [7, Theorem 3.3].

Proposition 2.26. *Let $A \in \mathcal{L}(Y, Z)$, $B \in \mathcal{L}(X, Y)$, and let $A_{M,N}^{(2)} \in \mathcal{L}(Z, Y)$, $B_{T,S}^{(2)} \in \mathcal{L}(Y, X)$ and $(AB)_{K,L}^{(2)} \in \mathcal{L}(Z, X)$ be outer inverses of A, B and AB with subspaces $K, T \subseteq X$, $M, S \subseteq Y$, and $N, L \subseteq Z$. Let operators $A'_{M,N} \in \mathcal{L}(Z, Y)$, $B'_{T,S} \in \mathcal{L}(Y, X)$ and $(AB)'_{K,L} \in \mathcal{L}(Z, X)$ satisfy*

$$\begin{aligned} R(A'_{M,N}) &= M, \quad N(A'_{M,N}) = N, \quad R(B'_{T,S}) = T, \quad N(B'_{T,S}) = S, \\ R(B'_{T,S}A'_{M,N}) &= R((AB)'_{K,L}) = K, \quad N(B'_{T,S}A'_{M,N}) = N((AB)'_{K,L}) = L. \end{aligned}$$

If $A_{M,N}^{(2)}A$ commutes with $BB'_{T,S}$ and $BB_{T,S}^{(2)}$ commutes with $A'_{M,N}A$, then

$$(AB)_{K,L}^{(2)} = B_{T,S}^{(2)}A_{M,N}^{(2)}.$$

For the (p, q) -outer generalized inverse we have the following result.

Proposition 2.27. *Let $a, w \in R$ and $e, f, p, q, k, l \in R^\bullet$ be such that $a_{p,q}^{(2)}$ and $w_{e,f}^{(2)}$ exist. If $a_{p,q}^{(2)}a$ commutes with $ww_{e,f}^{(2)}$, then $(aw)_{k,l}^{(2)}$ exists and $(aw)_{k,l}^{(2)} = w_{e,f}^{(2)}a_{p,q}^{(2)}$ if and only if $w_{e,f}^{(2)}pw = k$ and $a(1-f)a_{p,q}^{(2)} = 1-l$.*

Proof. Since $a_{p,q}^{(2)}$ commutes with $ww_{e,f}^{(2)}$, we have

$$w_{e,f}^{(2)}a_{p,q}^{(2)}(aw)w_{e,f}^{(2)}a_{p,q}^{(2)} = w_{e,f}^{(2)}(a_{p,q}^{(2)}aww_{e,f}^{(2)})a_{p,q}^{(2)} = w_{e,f}^{(2)}(ww_{e,f}^{(2)}a_{p,q}^{(2)}a)a_{p,q}^{(2)} = w_{e,f}^{(2)}a_{p,q}^{(2)}.$$

Also,

$$\begin{aligned} w_{e,f}^{(2)}a_{p,q}^{(2)}(aw) &= w_{e,f}^{(2)}(a_{p,q}^{(2)}a)w = w_{e,f}^{(2)}pw, \\ (aw)w_{e,f}^{(2)}a_{p,q}^{(2)} &= a(ww_{e,f}^{(2)})a_{p,q}^{(2)} = a(1-f)a_{p,q}^{(2)}. \end{aligned}$$

Therefore, by the definition of (p, q) -outer generalized inverse, we obtain the conclusion. \square

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